

Applications of Kronecker's limit formula for elliptic Eisenstein series

Jay Jorgenson

Anna-Maria von Pippich

Lejla Smajlović *

May 13, 2015

Abstract

We develop two applications of the Kronecker's limit formula associated to elliptic Eisenstein series: A factorization theorem for holomorphic modular forms, and a proof of Weil's reciprocity law. Several examples of the general factorization results are computed, specifically for certain moonshine groups, congruence subgroups, and, more generally, non-compact subgroups with one cusp. In particular, we explicitly compute the Kronecker limit function associated to certain elliptic points for a few small level moonshine groups.

1 Introduction and statement of results

1.1 Non-holomorphic Eisenstein series.

Let Γ be a Fuchsian group of the first kind which acts on the hyperbolic space \mathbb{H} by fractional linear transformations, and let $M = \Gamma \backslash \mathbb{H}$ be the finite volume quotient. One can view M as a finite volume hyperbolic Riemann surface, possibly with cusps and elliptic fixed points. In a slight abuse of notation, we will use M to denote both the Riemann surface as well as a (Ford) fundamental domain of Γ acting on \mathbb{H} .

The abelian subgroups of Γ are classified as three distinct types: Parabolic, hyperbolic and elliptic. Accordingly, there are three types of scalar-valued non-holomorphic Eisenstein series, whose definitions we now recall.

Parabolic subgroups are characterized by having a unique fixed point P on the extended upper-half plane $\hat{\mathbb{H}}$. The fixed point P is known as a cusp of M , and the associated parabolic subgroup is denoted by Γ_P . The parabolic Eisenstein series $\mathcal{E}_P^{\text{par}}(z, s)$ associated to P is defined for $z \in M$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, by the series

$$\mathcal{E}_P^{\text{par}}(z, s) = \sum_{\eta \in \Gamma_P \backslash \Gamma} \text{Im}(\sigma_P^{-1} \eta z)^s,$$

where σ_P is the scaling matrix for the cusp P , i.e. the element of $\text{PSL}_2(\mathbb{R})$ such that, when extending the action of σ_P to $\hat{\mathbb{H}}$, we have that $\sigma_P \infty = P$.

Hyperbolic subgroups have two fixed points on the extended upper-half plane $\hat{\mathbb{H}}$. Let us denote a hyperbolic subgroup by Γ_γ for $\gamma \in \Gamma$, and let \mathcal{L}_γ signify the geodesic path in \mathbb{H} connecting the two fixed points of hyperbolic element γ . Following Kudla and Millson from [KM79], one defines a scalar-valued hyperbolic Eisenstein series for $z \in M$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$ by the series

$$\mathcal{E}_\gamma^{\text{hyp}}(z, s) = \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \cosh(d_{\text{hyp}}(\eta z, \mathcal{L}_\gamma))^{-s}, \quad (1)$$

*The first named author acknowledges grant support from the NSF and PSC-CUNY. We thank Professor Floyd Williams for making available to us the unpublished dissertation [Va96] which was written by his student I. N. Vassileva. The results in the manuscript were of great interest to us, and we hope the document will become available to the mathematical community.

where $d_{\text{hyp}}(\eta z, \mathcal{L}_\gamma)$ is the hyperbolic distance from the point ηz to \mathcal{L}_γ .

Elliptic subgroups have finite order and have a unique fixed point within \mathbb{H} . In fact, for any point $w \in M$, there is an elliptic subgroup Γ_w which fixes w , where in all but a finite number of cases Γ_w is the identity element. Elliptic Eisenstein series were defined in an unpublished manuscript from 2004 by Jorgenson and Kramer and were studied in depth in the 2010 dissertation [vP10] by von Pippich. Specifically, for $z \in M$, $z \neq w$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, the elliptic Eisenstein series is defined by

$$\mathcal{E}_w^{\text{ell}}(z, s) = \sum_{\eta \in \Gamma_w \backslash \Gamma} \sinh(d_{\text{hyp}}(\eta z, w))^{-s} \quad (2)$$

where $d_{\text{hyp}}(\eta z, w)$ denotes the hyperbolic distance from ηz to w .

1.2 Known properties and relations

There are some fundamental differences between the three types of Eisenstein series defined above. Hyperbolic Eisenstein series are in $L^2(M)$, whereas parabolic and elliptic series are not. Elliptic Eisenstein series are defined as a sum over a finite index subset of Γ , and indeed the series (2) can be extended to all Γ which would introduce a multiplicative factor equal to the order of Γ_w . However, hyperbolic and parabolic series are necessarily formed by sums over infinite index subsets of Γ . Parabolic Eisenstein series are eigenfunctions of the hyperbolic Laplacian; however, elliptic and hyperbolic Eisenstein series satisfy a differential-difference equation which involves the value of the series at $s + 2$.

Despite their differences, there are several intriguing ways in which the Eisenstein series interact. Since the hyperbolic Eisenstein series are in $L^2(M)$, the expression (1) admits a spectral expansion which involves the parabolic Eisenstein series; see [JKvP10] and [KM79]. If one considers a degenerating sequence of Riemann surfaces obtained by pinching a geodesic, then the associated hyperbolic Eisenstein series converges to parabolic Eisenstein series on the limit surface; see [Fa07] and [GJM08]. If one studies a family of elliptically degenerating surfaces obtained by re-uniformizing at a point with increasing order, then the corresponding elliptic Eisenstein series converge to parabolic Eisenstein series on the limit surface; see [GvP09].

Finally, there are some basic similarities amongst the series. Each series admits a meromorphic continuation to all $s \in \mathbb{C}$. The poles of the meromorphic continuations have been identified and are closely related, in all cases involving data associated to the continuous and non-cuspidal discrete spectrum of the hyperbolic Laplacian and, for hyperbolic and elliptic series, involving data associated to the cuspidal spectrum as well. Finally, and most importantly for this article, the hyperbolic and elliptic Eisenstein series are holomorphic at $s = 0$, and for all known instances, the parabolic Eisenstein series also is holomorphic at $s = 0$. In all these cases, the value of each Eisenstein series at $s = 0$ is a constant as a function of z . The coefficient of s in the Taylor series expansion about $s = 0$ shall be called the Kronecker limit function.

1.3 Kronecker limit functions

The classical Kronecker's limit formula is the following statement, which we quote from [Si80]. If we consider the case when $\Gamma = \text{PSL}_2(\mathbb{Z})$, then

$$\mathcal{E}_\infty^{\text{par}}(z, s) = \frac{3}{\pi(s-1)} - \frac{1}{2\pi} \log(|\Delta(z)| \text{Im}(z)^6) + C + O(s-1) \quad \text{as } s \rightarrow 1,$$

where $C = 6(1 - 12\zeta'(-1) - \log(4\pi))/\pi$, and with Dedekind's delta function $\Delta(z)$ given by

$$\Delta(z) = \left[q_z^{1/24} \prod_{n=1}^{\infty} (1 - q_z^n) \right]^{24} = \eta(z)^{24} \quad \text{with } q_z = e^{2\pi iz}.$$

By employing the well-known functional equation for $\mathcal{E}_\infty^{\text{par}}(z, s)$, Kronecker's limit formula can be reformulated as

$$\mathcal{E}_\infty^{\text{par}}(z, s) = 1 + \log(|\Delta(z)|^{1/6} \text{Im}(z))s + O(s^2) \quad \text{as } s \rightarrow 0.$$

For general Fuchsian groups of the first kind, Goldstein [Go73] studied analogue of Kronecker's limit formula associated to parabolic Eisenstein series. We will use the results from [Go73] throughout this article.

The hyperbolic Eisenstein series in [KM79] are form-valued, and the series are defined by an infinite sum which converges for $\text{Re}(s) > 0$. The main result in [KM79] is that the form-valued hyperbolic Eisenstein series is holomorphic at $s = 0$, and the value is equal to the harmonic form that is the Poincaré dual to the one-cycle in the homology group $H^1(M, \mathbb{R})$ corresponding to the hyperbolic geodesic γ fixed by Γ_γ .

The analogue of Kronecker's limit formula for elliptic Eisenstein series was first proved in [vP10] and [vP15]. Specifically, it is shown that at $s = 0$, the series (2) admits the Laurent expansion

$$\begin{aligned} \text{ord}(w) \mathcal{E}_w^{\text{ell}}(z, s) - \frac{2^s \sqrt{\pi} \Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{k=1}^{p_\Gamma} \mathcal{E}_{p_k}^{\text{par}}(w, 1-s) \mathcal{E}_{p_k}^{\text{par}}(z, s) = \\ -c - \log(|H_\Gamma(z, w)|^{\text{ord}(w)} (\text{Im}(z))^c) \cdot s + O(s^2) \quad \text{as } s \rightarrow 0, \end{aligned} \quad (3)$$

where p_k , $k = 1, \dots, p_\Gamma$, are cusps of M , $c = 2\pi/\text{Vol}_{\text{hyp}}(M)$, and $H_\Gamma(z, w)$ is a holomorphic automorphic function with respect to Γ and which vanishes only when $z = \eta w$ for some $\eta \in \Gamma$. Two explicit computations are given in [vP10] and [vP15] for $\Gamma = \text{PSL}_2(\mathbb{Z})$ when considering the elliptic Eisenstein series $E_w^{\text{ell}}(z, s)$ associated to the points $w = i$ and $w = \rho = (1 + i\sqrt{3})/2$. In these cases, the elliptic Kronecker limit function $H_\Gamma(z, w)$ at points $w = i$ and $w = \rho$ is such that

$$|H_\Gamma(z, i)| = \exp(-B_i) |E_6(z)|, \quad \text{where } B_i = -3(24\zeta'(-1) - \log(2\pi) + 4 \log \Gamma(1/4)) \quad (4)$$

and

$$|H_\Gamma(z, \rho)| = \exp(-B_\rho) |E_4(z)|, \quad \text{where } B_\rho = -2(24\zeta'(-1) - 2 \log(2\pi/\sqrt{3}) + 6 \log \Gamma(1/3)). \quad (5)$$

The Kronecker limit formula for elliptic Eisenstein series became the asymptotic formulas

$$\mathcal{E}_i^{\text{ell}}(z, s) = -\log(|E_6(z)| |\Delta(z)|^{-1/2}) \cdot s + O(s^2) \quad \text{as } s \rightarrow 0, \quad (6)$$

and

$$\mathcal{E}_\rho^{\text{ell}}(z, s) = -\log(|E_4(z)| |\Delta(z)|^{-1/3}) \cdot s + O(s^2) \quad \text{as } s \rightarrow 0, \quad (7)$$

where E_4 and E_6 are classical holomorphic Eisenstein series on $\text{PSL}_2(\mathbb{Z})$ of weight four and six, respectively.

Before continuing, let us state what we believe to be an interesting side comment. The Kronecker limit function associated to elliptic Eisenstein series is naturally defined as the coefficient of s in the Laurent expansion of the elliptic Eisenstein series near $s = 0$. As we show below, one can realize the Kronecker limit function for parabolic Eisenstein series for groups with one cusp as the coefficient of s in the Laurent expansion of the parabolic Eisenstein series at $s = 0$. One has yet to study the Laurent expansion near $s = 0$, in particular the coefficient of s , for the scalar-valued hyperbolic Eisenstein series; for that matter, we have not fully understood the analogous question for the vector of parabolic Eisenstein series for general groups. We expect that one can develop systematic theory by focusing on coefficients of s in all cases.

1.4 Important comment and assumption

At this time, we do not have a complete understanding of the behavior of the parabolic Eisenstein series $\mathcal{E}_P^{\text{par}}(z, s)$ near $s = 0$. If the group has one cusp, the functional equation of the Eisenstein series shows that $\mathcal{E}_P^{\text{par}}(z, 0) = 1$. In notation to be set below, its scattering determinant is zero at $s = 0$. However, this is not true when there is more than one cusp. For example, on page 536 of [He83], the author computes the scattering matrix for $\Gamma_0(N)$ for square-free N , from which it is clear that $\Phi(s)$ is holomorphic but not zero at $s = 0$. Specifically, it remains to determine if the parabolic Eisenstein series is holomorphic at $s = 0$, which is a question we were unable to answer in complete generality.

Throughout this article, we assume that $\mathcal{E}_P^{\text{par}}(z, s)$ is holomorphic at $s = 0$.

The assumption is true in all the instances where specific examples are developed.

1.5 Main results

The purpose of the present paper is to further study the Kronecker limit function associated to elliptic Eisenstein series. We develop two applications. To begin, we examine the relation (3) and study the contribution near $s = 0$ of the term involving the parabolic Eisenstein series. As with the parabolic Eisenstein series, the resulting expression is particularly simple in the case when the group Γ has one cusp. However, in all cases, we obtain an asymptotic formula for $\mathcal{E}_w^{\text{ell}}(z, s)$ near $s = 0$ which allows us to prove asymptotic bounds for the elliptic Kronecker limit function in any parabolic cusp associated to Γ . As a consequence, we are able to prove the main result of this article, namely a factorization theorem which expresses holomorphic forms on M of arbitrary weight as products of the elliptic Kronecker limit functions.

The product formulas are developed in detail in the case of so-called moonshine groups, which are discrete groups obtained by adding the Fricke involutions to the congruence subgroups $\Gamma_0(N)$. As an application of the factorization theorem, we establish further examples of relations similar to (4), (5), (6) and (7). For example, the moonshine group $\Gamma = \overline{\Gamma_0(2)^+} = \Gamma_0(2)^+ / \{\pm \text{Id}\}$ has $e_2 = 1/2 + i/2$ as a fixed point of order four. In section 6.2, we prove that the elliptic Kronecker limit function $H_2(z, e_2)$ associated to the point e_2 is such that

$$|H_2(z, e_2)| = \exp(-B_{2, e_2}) \left| E_4^{(2)}(z) \right|^{1/2},$$

where $E_4^{(2)}(z)$ is the weight four holomorphic Eisenstein series associated to $\Gamma_0(2)^+$ and

$$B_{2, e_2} = - \left(24\zeta'(-1) + \log(8\pi^2) - \frac{11}{6} \log 2 + \frac{1}{12} \log(|\Delta(1/2 + i/2) \cdot \Delta(1 + i)|) \right).$$

In this case, the Kronecker limit formula for the elliptic Eisenstein series $\mathcal{E}_{e_2}^{\text{ell}}(z, s)$ reads as

$$\mathcal{E}_{e_2}^{\text{ell}}(z, s) = -\log \left(|E_4^{(2)}(z)|^{1/2} |\Delta(z)\Delta(2z)|^{-1/12} \right) \cdot s + O(s^2) \quad \text{as } s \rightarrow 0,$$

or, equivalently, as

$$\mathcal{E}_{e_2}^{\text{ell}}(z, s) = -\log \left(\frac{1}{\sqrt{5}} |E_4(z) + 4E_4(2z)|^{1/2} |\Delta(z)\Delta(2z)|^{-1/12} \right) \cdot s + O(s^2) \quad \text{as } s \rightarrow 0. \quad (8)$$

The factorization theorem allows one to formulate numerous of examples of this type, of which we develop a few for certain moonshine and congruence subgroups.

Second, we use the elliptic Kronecker limit formula to give a new proof of Weil's reciprocity formula. A number of authors have obtained generalizations of Weil's reciprocity law; see, for example, the elegant presentation in [Kh08] which discusses various reciprocity laws over \mathbb{C} as well as Deligne's article [De91] where the author re-interprets Tate's local symbol and obtains a number of generalizations and applications. It would be interesting to study the possible connection between the functional analytic method of the present and companion article [JvPS14] with the algebraic ideas in [De91] and results surveyed in [Kh08].

An outline of this article is as follows. In section 2 we establish notation and cite various results from the literature. In section 3, we reformulate Kronecker's limit formula for parabolic Eisenstein series as an asymptotic statement near $s = 0$. From the results in section 3, we then prove, in section 4, the asymptotic behavior in the cusps of the elliptic Kronecker limit function. Specific examples are given for moonshine groups $\overline{\Gamma_0(N)^+}$ with square-free level N and congruence subgroups $\overline{\Gamma_0(p)}$ with prime level p . In section 5 we prove the factorization theorem which states, in somewhat vague terms, that any holomorphic form on M can be written as a product of elliptic Kronecker limit functions, up to a multiplicative constant. In addition, from the asymptotic

formula from section 4, one is able to obtain specific information associated to the multiplicative constant in the aforementioned description of the factorization theorem. In section 6 we give examples of the factorization theorem for holomorphic Eisenstein series for the modular group, for moonshine groups of levels 2 and 5, for general moonshine groups, and for congruence subgroups $\overline{\Gamma_0(p)}$ of prime level. Finally, in section 7, we present our proof of Weil's reciprocity using the elliptic Kronecker limit functions and state a few concluding remarks.

2 Background material

2.1 Basic notation

Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ denote a Fuchsian group of the first kind acting by fractional linear transformations on the hyperbolic upper half-plane $\mathbb{H} := \{z = x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}; y > 0\}$. We let $M := \Gamma \backslash \mathbb{H}$, which is a finite volume hyperbolic Riemann surface, and denote by $p : \mathbb{H} \rightarrow M$ the natural projection. We assume that M has e_Γ elliptic fixed points and p_Γ cusps. We identify M locally with its universal cover \mathbb{H} .

We let μ_{hyp} denote the hyperbolic metric on M , which is compatible with the complex structure of M , and has constant negative curvature equal to minus one. The hyperbolic line element ds_{hyp}^2 , resp. the hyperbolic Laplacian Δ_{hyp} , are given as

$$ds_{\mathrm{hyp}}^2 := \frac{dx^2 + dy^2}{y^2}, \quad \text{resp.} \quad \Delta_{\mathrm{hyp}} := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

By $d_{\mathrm{hyp}}(z, w)$ we denote the hyperbolic distance from $z \in \mathbb{H}$ to $w \in \mathbb{H}$.

2.2 Moonshine groups

Let $N = p_1 \cdots p_r$ be a square-free, non-negative integer. The subset of $\mathrm{SL}_2(\mathbb{R})$, defined by

$$\Gamma_0(N)^+ := \left\{ e^{-1/2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : ad - bc = e, \ a, b, c, d, e \in \mathbb{Z}, \ e \mid N, \ e \mid a, \ e \mid d, \ N \mid c \right\}$$

is an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{R})$. We use the terminology ‘‘moonshine group’’ of level N to describe $\Gamma_0(N)^+$ because of the important role these groups play in ‘‘monstrous moonshine’’. Previously, the groups $\Gamma_0(N)^+$ were studied in [Hel66] where it was proved that if a subgroup $G \subseteq \mathrm{SL}_2(\mathbb{R})$ is commensurable with $\mathrm{SL}_2(\mathbb{Z})$, then there exists a square-free, non-negative integer N such that G is a subgroup of $\Gamma_0(N)^+$. We also refer to page 27 of [Sh71] where the groups $\Gamma_0(N)^+$ are cited as examples of groups which are commensurable with $\mathrm{SL}_2(\mathbb{Z})$ but non necessarily conjugate to a subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Let $\{\pm \mathrm{Id}\}$ denote the set of two elements consisting of the identity matrix Id and its product with -1 . In general, if Γ is a subgroup of $\mathrm{SL}_2(\mathbb{R})$, we let $\overline{\Gamma} := \Gamma / \{\pm \mathrm{Id}\}$ denote its projection into $\mathrm{PSL}_2(\mathbb{R})$.

2.3 Holomorphic Eisenstein series

Following [Se73], we define a weakly modular form f of weight $2k$ for $k \geq 1$ associated to Γ to be a function f which is meromorphic on \mathbb{H} and satisfies the transformation property

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^{-2k} f(z) \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Let Γ be a Fuchsian group of the first kind that has at least one class of parabolic elements. By rescaling, if necessary, we may always assume that the parabolic subgroup of Γ has a fixed point at ∞ , with identity scaling matrix. In this situation, any weakly modular form f will satisfy the

relation $f(z+1) = f(z)$, so we can write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q_z^n \quad \text{where } q_z = e(z) = e^{2\pi iz}.$$

If $a_n = 0$ for all $n < 0$, then f is said to be holomorphic at the cusp at ∞ .

A holomorphic modular form with respect to Γ is a weakly modular form which is holomorphic on \mathbb{H} and in all of the cusps of Γ . Examples of holomorphic modular forms are the holomorphic Eisenstein series, which are defined as follows. Let Γ_∞ denote the subgroup of Γ which stabilizes the cusp at ∞ . For $k \geq 2$, let

$$E_{2k,\Gamma}(z) := \sum_{\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma} (cz + d)^{-2k}. \quad (9)$$

It is elementary to show that the series on the right-hand side of (9) is absolutely convergent for all integers $k \geq 2$ and defines a holomorphic modular form of weight $2k$ with respect to Γ . Furthermore, the series $E_{2k,\Gamma}$ is bounded and non-vanishing at cusps and such that

$$E_{2k,\Gamma}(z) = 1 + O(\exp(-2\pi \operatorname{Im}(z))), \quad \text{as } \operatorname{Im}(z) \rightarrow \infty.$$

When $\Gamma = \operatorname{PSL}_2(\mathbb{Z})$, we denote $E_{2k,\operatorname{PSL}_2(\mathbb{Z})}$ by E_{2k} . The holomorphic forms $E_{2k}(z)$ have the q -expansions

$$E_{2k}(z) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q_z^n,$$

where B_{2k} denotes the $2k$ -th Bernoulli number and σ_l is the generalized divisor function, which is defined by $\sigma_l(m) = \sum_{d|m} d^l$. By convention, we set $\sigma(m) = \sigma_1(m)$.

On the full modular surface, there is no weight 2 holomorphic modular form. Consider, however, the function $E_2(z)$ defined by its q -expansion

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q_z^n$$

which transforms according to the formula

$$E_2(\gamma z) = (cz + d)^2 E_2(z) + \frac{6}{\pi i} c(cz + d),$$

for $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \operatorname{PSL}_2(\mathbb{Z})$. It is elementary to show that for a prime p , the function

$$E_{2,p}(z) := E_2(z) - pE_2(pz) \quad (10)$$

is a weight 2 holomorphic form associated to the congruence subgroup $\overline{\Gamma_0(p)}$ of $\operatorname{PSL}_2(\mathbb{Z})$. The q -expansion of $E_{2,p}$ is

$$E_{2,p}(z) = (1-p) - 24 \sum_{n=1}^{\infty} \sigma(n) (q_z^n - p q_z^{pn}). \quad (11)$$

When $\Gamma = \overline{\Gamma_0^+(N)}$, we denote the forms $E_{2k,\overline{\Gamma_0^+(N)}}$ by $E_{2k}^{(N)}$. In [JST14a] it is proved that $E_{2k}^{(N)}(z)$ may be expressed as a linear combination of forms $E_{2k}(z)$, with dilated arguments, namely

$$E_{2k}^{(N)}(z) = \frac{1}{\sigma_k(N)} \sum_{v|N} v^k E_{2k}(vz). \quad (12)$$

2.4 Scattering matrices

Assume that the surface M has p_Γ cusps, we let P_j with $j = 1, \dots, p_\Gamma$ denote the individual cusps. Denote by ϕ_{jk} , with $j, k = 1, \dots, p_\Gamma$, the entries of the hyperbolic scattering matrix $\Phi_M(s)$ which are computed from the constant terms in the Fourier expansion of the parabolic Eisenstein series $\mathcal{E}_{P_j}^{\text{par}}(z, s)$ associated to cusp P_j in an expansion in the cusp P_k . For all $j, k = 1, \dots, p_\Gamma$, each function ϕ_{jk} has a simple pole at $s = 1$ with residue equal to $1/\text{Vol}_{\text{hyp}}(M)$. Furthermore, ϕ_{jk} has a Laurent series expansion at $s = 1$ which we write as

$$\phi_{jk}(s) = \frac{1}{\text{Vol}_{\text{hyp}}(M)(s-1)} + \beta_{jk} + \gamma_{jk}(s-1) + O((s-1)^2), \text{ as } s \rightarrow 1. \quad (13)$$

After a slight renormalization and trivial generalization, Theorem 3-1 from [Go73] asserts that the parabolic Eisenstein series $\mathcal{E}_{P_j}^{\text{par}}(z, s)$ admits the Laurent expansion

$$\mathcal{E}_{P_j}^{\text{par}}(z, s) = \frac{1}{\text{Vol}_{\text{hyp}}(M)(s-1)} + \beta_{jj} - \frac{1}{\text{Vol}_{\text{hyp}}(M)} \log \left| \eta_{P_j}^4(z) \text{Im}(z) \right| + f_j(z)(s-1) + O((s-1)^2), \quad (14)$$

as $s \rightarrow 1$, for $j = 1, \dots, p_\Gamma$.

As the notation suggests, the function $\eta_{P_j}(z)$ is a holomorphic form for Γ and is a generalization of the classical eta function for the full modular group. To be precise, $\eta_{P_j}(z)$ is an automorphic form corresponding to the multiplier system $v(\sigma) = \exp(i\pi S_{\Gamma,j}(\sigma))$, where $S_{\Gamma,j}(\sigma)$ is a generalization of a Dedekind sum attached to a cusp P_j for each $j = 1, \dots, p_\Gamma$ of M , meaning a real number uniquely determined for every $\sigma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ which satisfies the relation

$$\log \eta_{P_j}(\sigma(z)) = \log \eta_{P_j}(z) + \frac{1}{2} \log(cz + d) + \pi i S_{\Gamma,j}(\sigma).$$

The coefficient $f_j(z)$ multiplying $(s-1)$ in formula (14) is a certain function, whose behavior is not of interest to us in this paper. This term would probably yield to a definition of generalized Dedekind sums; see, for example, [Ta86].

Finally, let us set the notation

$$\phi_{jk}(s) = a_{jk} + b_{jk}s + c_{jk}s^2 + O(s^3) \text{ as } s \rightarrow 0 \quad (15)$$

for the coefficients in the Laurent expansion of ϕ_{jk} near $s = 0$. Note that the form of this expansion is justified by the assumption made in subsection 1.4.

3 Kronecker's limit formula for parabolic Eisenstein series

In this section we will re-write the Kronecker limit formula for the parabolic Eisenstein series as an expression involving the Laurent expansion near $s = 0$. We begin with the following lemma which states certain relations amongst coefficients appearing in (13) and (15). *To repeat, we assume that each parabolic Eisenstein series $\mathcal{E}_{P_j}^{\text{par}}(z, s)$ is holomorphic at $s = 0$.*

Lemma 1. *With the notation in (13) and (15), we have, for each $k, l = 1, \dots, p_\Gamma$, the following relations:*

$$\sum_{j=1}^{p_\Gamma} a_{jk} = 0, \quad (16)$$

$$\sum_{j=1}^{p_\Gamma} \left(-\frac{b_{jk}}{\text{Vol}_{\text{hyp}}(M)} + a_{jk}\beta_{jl} \right) = \delta_{kl}, \quad (17)$$

$$\sum_{j=1}^{p_\Gamma} \left(-\frac{c_{jk}}{\text{Vol}_{\text{hyp}}(M)} + b_{jk}\beta_{jl} \right) = \sum_{j=1}^{p_\Gamma} a_{jk}\gamma_{jl}, \quad (18)$$

where δ_{kl} is the Kronecker symbol.

Proof. The relations (16) through (18) are immediate consequences of the functional equation for the scattering determinant, namely the formula $\Phi_M(s)\Phi_M(1-s) = \text{Id}$. In particular, the formulae are obtained by computing the coefficients of s^{-1} , 1, and s in the Laurent expansion near $s = 0$. \square

Proposition 2. *With the notation in (13) and (15), the parabolic Eisenstein series $\mathcal{E}_{P_j}^{\text{par}}(z, s)$ has a Taylor series expansion at $s = 0$ which can be written as*

$$\begin{aligned} \mathcal{E}_{P_j}^{\text{par}}(z, s) = & \sum_{k=1}^{p_\Gamma} \left[-\frac{b_{jk}}{\text{Vol}_{\text{hyp}}(M)} + a_{jk} \left(\beta_{kk} - \frac{1}{\text{Vol}_{\text{hyp}}(M)} \log |\eta_{P_k}^4(z) \text{Im } z| \right) \right] + \\ & + s \cdot \sum_{k=1}^{p_\Gamma} \left[-\frac{c_{jk}}{\text{Vol}_{\text{hyp}}(M)} + b_{jk} \left(\beta_{kk} - \frac{1}{\text{Vol}_{\text{hyp}}(M)} \log |\eta_{P_k}^4(z) \text{Im } z| \right) + a_{jk} f_k(z) \right] + O(s^2). \end{aligned} \quad (19)$$

Proof. The result is a straightforward computation based on the functional equation

$$(\mathcal{E}_1^{\text{par}}(z, s) \dots \mathcal{E}_p^{\text{par}}(z, s))^T = \Phi_M(s)(\mathcal{E}_1^{\text{par}}(z, 1-s) \dots \mathcal{E}_p^{\text{par}}(z, 1-s))^T$$

together with the expansions (14) and (15). \square

In the case when $p_\Gamma = 1$, the relations (16) through (18) and Proposition 2 become particularly simple and yield an elegant statement. As is standard, the cusp is normalized to be at ∞ , and the associated Eisenstein series, eta function, scattering coefficients, etc. are written with the subscript ∞ .

Corollary 3. *The Kronecker limit formula for parabolic Eisenstein series $\mathcal{E}_\infty^{\text{par}}$ on a finite volume Riemann surface with one cusp at ∞ can be written as*

$$\mathcal{E}_\infty^{\text{par}}(z, s) = 1 + \log(|\eta_\infty^4(z)| \text{Im}(z))s + O(s^2), \text{ as } s \rightarrow 0. \quad (20)$$

Example 4. In the case when $\Gamma = \overline{\Gamma_0^+(N)}$, for a square-free, positive integer N , the quotient space $X_N := \overline{\Gamma_0^+(N)} \backslash \mathbb{H}$ has one cusp. The automorphic form η_∞ is explicitly computed in [JST14a], where it is proved that

$$\eta_\infty(z) = \sqrt[2^r]{\prod_{v|N} \eta(vz)}.$$

Example 5. In the case when Γ is the group $\overline{\Gamma_0(N)}$, for a positive integer N , the corresponding quotient space $M_N := \overline{\Gamma_0(N)} \backslash \mathbb{H}$ has many cusps. Using a standard fundamental domain, M_N has cusps at ∞ , at 0 and, in the case when N is not prime, at the rational points $1/v$, where $v \mid N$ is such that $(v, \frac{N}{v}) = 1$, where (\cdot, \cdot) stands for the greatest common divisor. As in the above example, let use the subscript ∞ to denote data associated to the cusp at ∞ . In particular, the automorphic form η_∞ in the example under consideration was explicitly computed in [Va96], where it is proved that

$$\eta_\infty(z) = \sqrt[\varphi(N)]{\prod_{v|N} \eta(vz)^{v\mu(N/v)}},$$

where $\varphi(N)$ is the Euler φ -function and μ denotes the Möbius function. In the case of other cusps P_k , the automorphic form η_{P_k} was also computed in [Va96], but the expressions are more involved so we omit repeating the formulas here.

Also, for the cusp at ∞ and the principal congruence subgroup $\Gamma(N)$, the eta-function is computed in Theorem 1, page 405 of [Ta86].

4 Kronecker's limit formula for elliptic Eisenstein series

The function $H_\Gamma(z, w)$, defined in (3) is called the *elliptic Kronecker limit function at w* . It satisfies the transformation rule

$$H_\Gamma(\gamma z, w) = \varepsilon_w(\gamma)(cz + d)^{2C_w} H_\Gamma(z, w), \text{ for any } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma, \quad (21)$$

where $\varepsilon_w(\gamma) \in \mathbb{C}$ is a constant of absolute value 1, independent of z and

$$C_w = 2\pi / (\text{ord}(w) \text{Vol}_{\text{hyp}}(M)), \quad (22)$$

see [vP10], Proposition 6.1.2., or [vP15]. Since $H_\Gamma(z, w)$, as a function of z , is finite and non-zero at the cusp $P_1 = \infty$, we may re-scale the function and assume, without loss of generality, that $H_\Gamma(z, w)$ is real at the cusp ∞ .

We begin by studying the asymptotic behavior of $H_\Gamma(\sigma_{P_l} z, w)$ as $y = \text{Im}(z) \rightarrow \infty$, for $l = 1, \dots, p_\Gamma$.

Proposition 6. *For any cusp P_l , with $l = 1, \dots, p_\Gamma$, let*

$$B_{w, P_l} = -C_w (2 - \log 2 + \log |\eta_{P_l}^4(w) \text{Im}(w)| - \beta_{ll} \text{Vol}_{\text{hyp}}(M)). \quad (23)$$

Then there exists a constant $a_{w, P_l} \in \mathbb{C}$ of modulus one such that

$$H(\sigma_{P_l} z, w) = a_{w, P_l} \exp(-B_{w, P_l}) |c_l z + d_l|^{2C_w} + O(\exp(-2\pi \text{Im}(z))), \text{ as } \text{Im}(z) \rightarrow \infty,$$

*where $\sigma_{P_l} = \begin{pmatrix} * & * \\ c_l & d_l \end{pmatrix}$ is the scaling matrix for a cusp P_l and C_w is defined by (22).*

Proof. The proof closely follows the proof of [vP10], Proposition 6.2.2. when combined with the Taylor series expansion (19) of the parabolic Eisenstein series at $s = 0$. For the convenience of the reader, we now present the complete argument.

Combining the equation (3) with the proof of Proposition 6.1.1 from [vP10], taking $e_j = w$, we can write

$$-\log(|H_\Gamma(z, w)| \text{Im}(z)^{C_w}) = \mathcal{K}_w(z),$$

where the function $\mathcal{K}_w(z)$ can be expressed as the sum of two terms: A term $\mathcal{F}_w(z)$ arising from the spectral expansion and a term $\mathcal{G}_w(z)$ which can be expressed as the sum over the group. Furthermore, for $z \in \mathbb{H}$ such that $\text{Im } z > \text{Im}(\gamma w)$ for all $\gamma \in \Gamma$ the parabolic Fourier expansion of $\mathcal{K}_w(\sigma_{P_l} z)$ is given by

$$\mathcal{K}_w(\sigma_{P_l} z) = \sum_{m \in \mathbb{Z}} b_{m, w, P_l}(y) e(mx)$$

with coefficients $b_{m, w, P_l}(y)$ given by

$$b_{m, w, P_l}(y) = \int_0^1 \mathcal{K}_w(\sigma_{P_l} z) e(-mx).$$

Since the hyperbolic Laplacian is SL_2 -invariant, we easily generalize computations from p. 128 of [vP10] to deduce that

$$\mathcal{K}_w(\sigma_{P_l} z) = -C_w \log y + A_{w, P_l} y + B_{w, P_l} + \sum_{m=1}^{\infty} (A_{m; w, P_l} e(mz) + \overline{A}_{m; w, P_l} e(-m\bar{z})),$$

for some constants $A_{w, P_l}, B_{w, P_l} \in \mathbb{R}$ and complex constants $A_{m; w, P_l}$.

Let us introduce the notation

$$f_{w, P_l}(z) := \exp \left(-2 \sum_{m=1}^{\infty} A_{m; w, P_l} e(mz) \right), \quad (24)$$

from which one immediately can write

$$\mathcal{K}_w(\sigma_{P_l} z) = A_{w,P_l} y + B_{w,P_l} - \log(|f_{w,P_l}(z)| \operatorname{Im}(z)^{C_w}). \quad (25)$$

When employing (25), we can re-write (3) as

$$\begin{aligned} \mathcal{E}_w^{\text{ell}}(\sigma_{P_l} z, s) - h_w(s) \sum_{j=1}^{p_\Gamma} \mathcal{E}_{P_j}^{\text{par}}(w, 1-s) \mathcal{E}_{P_j}^{\text{par}}(\sigma_{P_l} z, s) = \\ - C_w + (A_{w,P_l} y + B_{w,P_l} - \log(|f_{w,P_l}(z)| \operatorname{Im}(z)^{C_w})) \cdot s + O(s^2), \end{aligned} \quad (26)$$

as $s \rightarrow 0$, where

$$h_w(s) := \frac{2^s \sqrt{\pi} \Gamma(s-1/2)}{\operatorname{ord}(w) \Gamma(s)}. \quad (27)$$

As in [vP10], pp. 129–130, we use the functional equation of the parabolic Eisenstein series and consider the constant term in the Fourier series expansion, as a function of z , of the function

$$\mathcal{E}_w^{\text{ell}}(\sigma_{P_l} z, s) - h_w(s) \sum_{j=1}^{p_\Gamma} \mathcal{E}_{P_j}^{\text{par}}(w, 1-s) \mathcal{E}_{P_j}^{\text{par}}(\sigma_{P_l} z, s) = \mathcal{E}_w^{\text{ell}}(\sigma_{P_l} z, s) - h_w(s) \sum_{j=1}^{p_\Gamma} \mathcal{E}_{P_j}^{\text{par}}(w, s) \mathcal{E}_{P_j}^{\text{par}}(\sigma_{P_l} z, 1-s). \quad (28)$$

The constant term is given by

$$-h_w(s) \sum_{j=1}^{p_\Gamma} \phi_{jl}(1-s) y^s \mathcal{E}_{P_j}^{\text{par}}(w, s) = -\frac{\sqrt{\pi}}{\operatorname{ord}(w)} \frac{\Gamma(s-1/2)}{\Gamma(s)} (2y)^s \sum_{j=1}^{p_\Gamma} \phi_{jl}(1-s) \mathcal{E}_{P_j}^{\text{par}}(w, s).$$

Recall the expansions

$$\Gamma(s-1/2) = -2\sqrt{\pi} (1 + (2 - \gamma - 2 \log 2)s + O(s^2)), \quad (29)$$

$$\frac{1}{\Gamma(s)} = s (1 + \gamma s + O(s^2)), \text{ and } (2y)^s = 1 + s \log(2y) + O(s^2), \quad (30)$$

which hold when $s \rightarrow 0$, where, as usual, γ denotes the Euler constant. When combining these expressions with (13), we can write the asymptotic expansions near $s = 0$ of the constant term in the Fourier series expansion of (28) as

$$\frac{2\pi}{\operatorname{ord}(w)} (1 + (2 + \log y - \log 2)s + O(s^2)) \cdot \sum_{j=1}^{p_\Gamma} \left(-\frac{1}{\operatorname{Vol}_{\text{hyp}}(M)} + \beta_{jl}s + O(s^2) \right) \mathcal{E}_{P_j}^{\text{par}}(w, s). \quad (31)$$

Let us now compute the first two terms in the Taylor series expansion at $s = 0$ of the expression

$$\sum_{j=1}^{p_\Gamma} \left(-\frac{1}{\operatorname{Vol}_{\text{hyp}}(M)} + \beta_{jl}s + O(s^2) \right) \mathcal{E}_{P_j}^{\text{par}}(w, s). \quad (32)$$

By applying (19), we conclude that the constant term in the Taylor series expansion of (32) is

$$\sum_{j=1}^{p_\Gamma} \sum_{k=1}^{p_\Gamma} \frac{-1}{\operatorname{Vol}_{\text{hyp}}(M)} \left(-\frac{b_{jk}}{\operatorname{Vol}_{\text{hyp}}(M)} + a_{jk} \beta_{kk} - \frac{a_{jk}}{\operatorname{Vol}_{\text{hyp}}(M)} \log |\eta_{P_k}^4(w) \operatorname{Im}(w)| \right).$$

Applying relations (16) and (17) we then obtain, by manipulation of the sums, that the constant term in (32) is equal to $-1/\operatorname{Vol}_{\text{hyp}}(M)$. The factor multiplying s is equal to

$$\begin{aligned} \sum_{j=1}^{p_\Gamma} \sum_{k=1}^{p_\Gamma} \frac{-1}{\operatorname{Vol}_{\text{hyp}}(M)} \left(-\frac{c_{jk}}{\operatorname{Vol}_{\text{hyp}}(M)} + b_{jk} \beta_{kk} - \frac{b_{jk}}{\operatorname{Vol}_{\text{hyp}}(M)} \log |\eta_{P_k}^4(w) \operatorname{Im}(w)| + a_{jk} f_k(w) \right) \\ + \sum_{j=1}^{p_\Gamma} \sum_{k=1}^{p_\Gamma} \beta_{jl} \left(-\frac{b_{jk}}{\operatorname{Vol}_{\text{hyp}}(M)} + a_{jk} \beta_{kk} - \frac{a_{jk}}{\operatorname{Vol}_{\text{hyp}}(M)} \log |\eta_{P_k}^4(w) \operatorname{Im}(w)| \right). \end{aligned}$$

Applying relations (16) to (18) we get that

$$\sum_{j=1}^{p_\Gamma} \sum_{k=1}^{p_\Gamma} a_{jk} f_k(w) = 0$$

and

$$\begin{aligned} & \sum_{k=1}^{p_\Gamma} \left(\frac{-1}{\text{Vol}_{\text{hyp}}(M)} \log |\eta_{P_k}^4(w) \text{Im}(w)| + \beta_{kk} \right) \sum_{j=1}^{p_\Gamma} \left(-\frac{b_{jk}}{\text{Vol}_{\text{hyp}}(M)} + a_{jk} \beta_{jl} \right) \\ &= \frac{-1}{\text{Vol}_{\text{hyp}}(M)} \log |\eta_{P_l}^4(w) \text{Im}(w)| + \beta_{ll} \end{aligned}$$

as well as

$$\sum_{j=1}^{p_\Gamma} \sum_{k=1}^{p_\Gamma} \left(\frac{-c_{jk}}{\text{Vol}_{\text{hyp}}(M)} + b_{jk} \beta_{jl} \right) = \sum_{j=1}^{p_\Gamma} \sum_{k=1}^{p_\Gamma} a_{jk} \gamma_{jl} = 0.$$

Therefore, the factor multiplying s in the Taylor series expansion of (32) is equal to

$$\frac{-1}{\text{Vol}_{\text{hyp}}(M)} \log |\eta_{P_l}^4(w) \text{Im}(w)| + \beta_{ll}.$$

Inserting this into (31) we see that the constant term in the Fourier series expansion of (28) is given by

$$-C_w - C'_w (2 - \log 2 + \log y + \log |\eta_{P_l}^4(w) \text{Im}(w)| - \beta_{ll} \text{Vol}_{\text{hyp}}(M)) s + O(s^2),$$

as $s \rightarrow 0$. Comparing this result with the right-hand side of formula (26), having in mind the definition of the number C_w , we immediately deduce that $A_{w,P_l} = 0$,

$$B_{w,P_l} = -C_w (2 - \log 2 + \log |\eta_{P_l}^4(w) \text{Im}(w)| - \beta_{ll} \text{Vol}_{\text{hyp}}(M))$$

and

$$\mathcal{K}_w(\sigma_{P_l} z) = -\log(|H_\Gamma(\sigma_{P_l} z, w)| |c_l z + d_l|^{-2C_w} \text{Im}(z)^{C_w}) = B_{w,P_l} - \log(|f_{w,P_l}(z)| \text{Im}(z)^{C_w}),$$

where the function f_{w,P_l} is defined by (24). From (24) we deduce that

$$|f_{w,P_l}(z)| = \exp \left(-2 \text{Re} \left(\sum_{m=1}^{\infty} A_{m;w,P_l} e(mz) \right) \right) = 1 + O(\exp(-2\pi \text{Im}(z))),$$

as $\text{Im}(z) \rightarrow \infty$. Therefore,

$$|H_\Gamma(\sigma_{P_l} z, w)| = \exp(-B_{w,P_l}) |c_l z + d_l|^{2C_w} + O(\exp(-2\pi \text{Im}(z))), \text{ as } \text{Im}(z) \rightarrow \infty,$$

and the proof is complete. \square

Example 7. Moonshine groups. Let $N = p_1 \cdots p_r$ be a squarefree number. Let $X_N = \overline{\Gamma_0(N)^+} \backslash \mathbb{H}$. The surface X_N possesses one cusp at ∞ with identity scaling matrix. The scattering determinant φ_N associated to the only cusp of X_N at ∞ is computed in [JST14], where it was shown that

$$\varphi_N(s) = \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)} \cdot D_N(s),$$

where $\zeta(s)$ is the Riemann zeta function and

$$D_N(s) = \prod_{j=1}^r \frac{p_j^{1-s} + 1}{p_j^s + 1} = \frac{1}{N^{s-1}} \prod_{j=1}^r \frac{p_j^{s-1} + 1}{p_j^s + 1}.$$

Let b_N denote the constant term in the Laurent series expansion of $\varphi_N(s)$ at $s = 1$. One can compute b_N by expanding functions $D_N(s)$, $\Gamma(s)$ and $\zeta(s)$ in their Laurent expansions at $s = 1$, which would yield the expressions

$$D_N(s) = \frac{2^r}{\sigma(N)} \left(1 + (s-1) \left(\sum_{j=1}^r \frac{(1-p_j) \log p_j}{2(p_j+1)} - \log N \right) + O((s-1)^2) \right),$$

and

$$\sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} = \pi (1 - 2 \log 2(s-1) + O((s-1)^2)), \quad (33)$$

as well as

$$\frac{\zeta(2s-1)}{\zeta(2s)} = \frac{6}{\pi^2} \left(\frac{1}{2(s-1)} - \log(2\pi) + 1 - 12\zeta'(-1) + O(s-1) \right). \quad (34)$$

Multiplying expansions (33) and (34) and using that

$$\frac{1}{\text{Vol}_{\text{hyp}}(X_N)} = \frac{3 \cdot 2^r}{\pi \sigma(N)},$$

which was proved in [JST14a], we arrive at the expression

$$b_N = -\frac{1}{\text{Vol}_{\text{hyp}}(X_N)} \left(\sum_{j=1}^r \frac{(p_j-1) \log p_j}{2(p_j+1)} - \log N + 2 \log(4\pi) + 24\zeta'(-1) - 2 \right). \quad (35)$$

With this formula, Proposition 6, and Example 4 we conclude that the elliptic Kronecker limit function $H_N(z, w) := H_{\Gamma_0^+(N)}(z, w)$ associated to the point $w \in X_N$ may be written as

$$H_N(z, w) = a_{N,w} \exp(-B_{N,w}) + \exp(-2\pi \text{Im}(z)), \text{ as } \text{Im}(z) \rightarrow \infty,$$

where $a_{N,w}$ is a complex constant of modulus one and

$$B_{N,w} = -\frac{2\pi}{\text{ord}(w) \text{Vol}_{\text{hyp}}(X_N)} \left(\sum_{j=1}^r \frac{(p_j-1) \log p_j}{2(p_j+1)} - \log N + C + \log \left(\sqrt[2^r]{\prod_{v|N} |\eta(vw)|^4} \cdot \text{Im}(w) \right) \right)$$

with $C := \log(8\pi^2) + 24\zeta'(-1)$.

Example 8. Congruence subgroups of prime level. Let $M_p = \overline{\Gamma_0(p)} \backslash \mathbb{H}$, where p is a prime. The surface M_p has two cusps, at ∞ and 0. The scaling matrix for the cusp at ∞ is identity matrix. The scattering matrix in this setting is computed in [He83] and is given by

$$\Phi_{M_p}(s) = \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)} \cdot \frac{1}{p^{2s}-1} \begin{pmatrix} p-1 & p^s - p^{1-s} \\ p^s - p^{1-s} & p-1 \end{pmatrix}.$$

Using the expansions (33) and (34), together with $\text{Vol}_{\text{hyp}}(M_p) = \pi(p+1)/3$ and the expansion

$$\frac{p-1}{p^{2s}-1} = \frac{1}{p+1} - \frac{2p^2 \log p}{(p-1)(p+1)^2} (s-1) + O((s-1)^2) \text{ as } s \rightarrow 1,$$

we conclude that the coefficients β_{11} and β_{22} in the Laurent series expansion (13) are given by

$$\beta_{11} = \beta_{22} = -\frac{2}{\text{Vol}_{\text{hyp}}(M_p)} \left(\log(4\pi p) + 12\zeta'(-1) - 1 + \frac{\log p}{p^2-1} \right).$$

Therefore, from Proposition 6, when applied to the cusp at ∞ , and Example 5, we conclude that the elliptic Kronecker limit function $\tilde{H}_p(z, w) := H_{\frac{p}{\Gamma_0(p)}}(z, w)$ associated to the point $w \in M_p$ can be written as

$$\tilde{H}_p(z, w) = \tilde{a}_{p,w} \exp(-\tilde{B}_{p,w}) + \exp(-2\pi \operatorname{Im}(z)), \text{ as } \operatorname{Im}(z) \rightarrow \infty,$$

where $\tilde{a}_{p,w}$ is a complex constant of modulus one and

$$\tilde{B}_{p,w} = -\frac{2\pi}{\operatorname{ord}(w) \operatorname{Vol}_{\text{hyp}}(M_p)} \left(\frac{2p^2 \log p}{p^2 - 1} + C + \log \left(\left| \sqrt[p-1]{\frac{\eta(pw)^p}{\eta(w)}} \cdot \operatorname{Im}(w) \right| \right) \right)$$

with $C := \log(8\pi^2) + 24\zeta'(-1)$.

5 A factorization theorem

In (4) and (5) one has an evaluation of the elliptic Kronecker limit function in the special case when $\Gamma = \operatorname{PSL}_2(\mathbb{Z})$ and $w = i$ or $w = \rho = \exp(2\pi i/3)$ are the elliptic fixed points of $\operatorname{PSL}_2(\mathbb{Z})$. The following theorem generalizes these results.

Theorem 9. *Let $M = \Gamma \backslash \mathbb{H}$ be a finite volume Riemann surface with at least one cusp, which we assume to be at ∞ with identity scaling matrix. Let k be a fixed positive integer such that there exists a weight $2k$ holomorphic form f_{2k} on M which is non-vanishing in all cusps and with q -expansion at ∞ given by*

$$f_{2k}(z) = b_{f_{2k}} + \sum_{n=1}^{\infty} b_{f_{2k}}(n) q_z^n. \quad (36)$$

Let $Z(f_{2k})$ denote the set of all zeros f_{2k} counted according to their multiplicities and let us define the function

$$H_{f_{2k}}(z) := \prod_{w \in Z(f_{2k})} H_{\Gamma}(z, w),$$

where, as above, $H_{\Gamma}(z, w)$ is the elliptic Kronecker limit function. Then there exists a complex constant $c_{f_{2k}}$ such that

$$f_{2k}(z) = c_{f_{2k}} H_{f_{2k}}(z) \quad (37)$$

and

$$|c_{f_{2k}}| = |b_{f_{2k}}| \exp \left(\sum_{w \in Z(f_{2k})} B_{w,\infty} \right),$$

where $B_{w,\infty}$ is defined in (23).

Proof. Assume that f_{2k} possesses $m + l \geq 1$ zeroes on M , where m zeros are at the elliptic points e_j of M , $j = 1, \dots, m$, and l zeroes are at the non-elliptic points $w_i \in M$; of course, all zeroes are counted with multiplicities. Then $H_{f_{2k}}(z)$ is a holomorphic function on M which is vanishing if and only if $z \in Z(f_{2k})$ and which according to (21) satisfies the transformation rule

$$H_{f_{2k}}(\gamma z) = \varepsilon_{f_{2k}}(\gamma) (cz + d)^{C_{f_{2k}}} H_{f_{2k}}(z), \text{ for any } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma,$$

where $\varepsilon_{f_{2k}}(\gamma)$ is a constant of modulus one and

$$C_{f_{2k}} = \frac{4\pi}{\operatorname{Vol}_{\text{hyp}}(M)} \left(\sum_{j=1}^m \frac{1}{n_{e_j}} + l \right).$$

The classical Riemann-Roch theorem relates the number of zeros of a holomorphic form to its weight and the genus of M in the case M is smooth and compact. A generalization of the relation

follows from Proposition 7, page II-7, of [SCM66] which, in the case under consideration, yields the formula

$$k \cdot \frac{\text{Vol}_{\text{hyp}}(M)}{2\pi} = \sum_{e \in \mathcal{E}_N} \frac{1}{n_e} v_e(f) + \sum_{z \in M \setminus \mathcal{E}_N} v_z(f), \quad (38)$$

where \mathcal{E}_N denotes the set of elliptic points in M , n_e is the order of the elliptic point $e \in \mathcal{E}_N$ and $v_z(f)$ denotes the order of the zero z of f .

Since $Z(f_{2k})$ is the set of all vanishing points of f_{2k} , formula (38) implies that

$$2k \cdot \frac{\text{Vol}_{\text{hyp}}(M)}{4\pi} = \sum_{j=1}^m \frac{1}{n_{e_j}} + l,$$

hence $C_{f_{2k}} = 2k$. In other words, $H_{f_{2k}}(z)$ is a holomorphic function on M , vanishing if and only if $z \in Z(f_{2k})$ and satisfying transformation rule

$$H_{f_{2k}}(\gamma z) = \varepsilon_{f_{2k}}(\gamma)(cz + d)^{2k} H_{f_{2k}}(z), \text{ for any } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma.$$

By Proposition 6, we have that for any $w \in Z(f_{2k})$ and any cusp P_l of M , with $l = 1, \dots, p_\Gamma$, the function

$$F_{f_{2k}}(z) := \frac{H_{f_{2k}}(z)}{f_{2k}(z)}$$

is a non-vanishing holomorphic function on M , bounded and non-zero at the cusp at ∞ and has at most polynomial growth in any other cusp of M . Therefore, the function $\log |F_{f_{2k}}(z)|$ is harmonic on M whose growth in any cusp is such that $\log |F_{f_{2k}}(z)|$ is L^2 on M . As a result, $\log |F_{f_{2k}}(z)|$ admits a spectral expansion; see [He83] or [Iwa02]. Since $\log |F_{f_{2k}}(z)|$ is harmonic, one can use integration by parts to show that $\log |F_{f_{2k}}(z)|$ is orthogonal to any eigenfunction of the Laplacian. Therefore, from the spectral expansion, one concludes that $\log |F_{f_{2k}}(z)|$ is constant, hence so is $F_{f_{2k}}(z)$. The evaluation of the constant is obtained by considering the limiting behavior as z approaches ∞ . With all this, the proof of (37) is complete. \square

6 Examples of factorization

6.1 An arbitrary surface with one cusp

In the case when a surface M has one cusp, we get the following special case of Theorem 9.

Corollary 10. *Let $M = \Gamma \backslash \mathbb{H}$ be a finite volume Riemann surface with one cusp, which we assume to be at ∞ with identity scaling matrix. Then the weight $2k$ holomorphic Eisenstein series $E_{2k,\Gamma}$ defined in (9) can be represented as*

$$E_{2k,\Gamma}(z) = a_{E_{2k,\Gamma}} B_{E_{2k,\Gamma}} \prod_{w \in Z(E_{2k,\Gamma})} H_\Gamma(z, w),$$

where $a_{E_{2k,\Gamma}}$ is a complex constant of modulus one and

$$B_{E_{2k,\Gamma}} = \prod_{w \in Z(E_{2k,\Gamma})} \exp(C_w (\log 2 - 2 + \beta_M \text{Vol}_{\text{hyp}}(M))) \cdot |\eta_\infty^4(w) \text{Im}(w)|^{-C_w}.$$

As before, η_∞ is the parabolic Kronecker limit function defined in section 3, formula (14), and β_M is the constant term in the Laurent series expansion of the scattering determinant on M .

In this case, due to a very simple form of the Kronecker limit formula for parabolic Eisenstein series as $s \rightarrow 0$, the factorization theorem yields an interesting form of the Kronecker limit formula for elliptic Eisenstein series, which we state as the following proposition.

Proposition 11. *Let $M = \Gamma \backslash \mathbb{H}$ be a finite volume Riemann surface with one cusp, which we assume to be at ∞ with identity scaling matrix. Let k be a fixed positive integer such that there exists a weight $2k$ holomorphic form f_{2k} on M with q -expansion at ∞ given by (36). Then*

$$\sum_{w \in Z(f_{2k})} \mathcal{E}_w^{\text{ell}}(z, s) = -s \log(|f_{2k}(z)| |\eta_\infty^4(z)|^{-k}) + s \log |b_{f_{2k}}| + O(s^2) \quad (39)$$

as $s \rightarrow 0$, where $Z(f_{2k})$ denotes the set of all zeros of f_{2k} counted with multiplicities.

Proof. We start with formula (3), which we divide by $\text{ord}(w)$, and take the sum over all $w \in Z(f_{2k})$ to get

$$\begin{aligned} \sum_{w \in Z(f_{2k})} \mathcal{E}_w^{\text{ell}}(z, s) - \mathcal{E}_\infty^{\text{par}}(z, s) &= \sum_{w \in Z(f_{2k})} h_w(s) \mathcal{E}_\infty^{\text{par}}(w, 1-s) = \\ &= \sum_{w \in Z(f_{2k})} C_w (1 + s \log(\text{Im } z)) - \log \left(\prod_{w \in Z(f_{2k})} |H_\Gamma(z, w)| \right) \cdot s + O(s^2) \end{aligned} \quad (40)$$

as $s \rightarrow 0$, where C_w and h_w are defined by (22) and (27) respectively. One now expands the second term on the left hand side of (40) into a Taylor series at $s = 0$ by applying formulas (29), (30), (20) and (14). After multiplication, we get, as $s \rightarrow 0$, the expression

$$\begin{aligned} \mathcal{E}_\infty^{\text{par}}(z, s) &= \sum_{w \in Z(f_{2k})} h_w(s) \mathcal{E}_\infty^{\text{par}}(w, 1-s) \\ &= \sum_{w \in Z(f_{2k})} C_w (1 + s [2 - \log 2 - \beta_M \text{Vol}(M) + \log |\eta_\infty^4(w) \text{Im}(w)| + |\eta_\infty^4(z) \text{Im}(z)|]) + O(s^2) \end{aligned} \quad (41)$$

as $s \rightarrow 0$. Theorem 9 yields that

$$\log \left(\prod_{w \in Z(f_{2k})} |H_\Gamma(z, w)| \right) = \log |f_{2k}(z)| - \sum_{w \in Z(f_{2k})} B_{w, \infty} - \log |b_{f_{2k}}|, \quad (42)$$

where $B_{w, \infty}$ is defined by (23) for the cusp $P_l = \infty$. Finally, from formula (38), we get that

$$\sum_{w \in Z(f_{2k})} C_w = k.$$

Therefore, by inserting (23), (42) and (41) and into (40), we immediately deduce (39). The proof is complete. \square

Remark 12. In the case $\Gamma = \text{PSL}_2(\mathbb{Z})$, the parabolic Kronecker limit function is given by $\eta_\infty(z) = \eta(z) = \Delta(z)^{1/24}$. Then, for $k = 3$ and $f_{2k} = E_6$, we have $b_{E_6} = 1$ and $Z(E_6) = \{i\}$, hence Proposition 11 yields (6). Analogously, for $k = 2$ and $f_{2k} = E_4$, we have $b_{E_4} = 1$ and $Z(E_4) = \{\rho\}$, and Proposition 11 gives (7). Furthermore (see [vP10], p. 131), we have $B_{E_6, \Gamma} = \exp(B_i)$ and $B_{E_4, \Gamma} = \exp(B_\rho)$, where B_i and B_ρ are given by (4) and (5) respectively.

Let us now develop further examples of a surfaces with one cusp and explicitly compute the constant $B_{E_{2k}, \Gamma}$ in these special cases.

6.2 Moonshine groups of square-free level

Example 13. Consider the surface X_2 . There exists one elliptic point of order two, $e_1 = i/\sqrt{2}$, and one elliptic point of order four, $e_2 = 1/2 + i/2$. The surface X_2 has genus zero and one cusp,

hence $\text{Vol}_{\text{hyp}}(X_2) = \pi/2$. The transformation rule for $E_6^{(2)}$ implies that the form must vanish at the points e_1 and e_2 . Furthermore, formula (38) when applied to X_2 becomes

$$\frac{2k}{8} = v_\infty(f) + \frac{1}{4}v_{e_2}(f) + \frac{1}{2}v_{e_1}(f) + \sum_{z \in X_2 \setminus \{e_1, e_2\}} v_z(f). \quad (43)$$

Taking $k = 3$, we conclude that e_1 and e_2 are the only vanishing points of $E_6^{(2)}$ and the order of vanishing is one at each point. Therefore, in the notation of Theorem 9 and Example 7, we have that the form $H_6^{(2)}(z) = H_{E_6^{(2)}}(z)$ is given by $H_6^{(2)}(z) := H_2(z, e_1)H_2(z, e_2)$. Assuming that the phase of $H_6^{(2)}(z)$ is such that it attains real values at the cusp ∞ , we have that

$$E_6^{(2)}(z) = C_{2,6}H_6^{(2)}(z), \quad (44)$$

where the absolute value of the constant $C_{2,6}$ is given by $|C_{2,6}| = e^{B_{2,e_1} + B_{2,e_2}}$ with

$$B_{2,e_1} = -2 \left(24\zeta'(-1) + \log(8\pi^2) - \frac{4}{3}\log 2 + \frac{1}{12}\log \left(\left| \Delta(i\sqrt{2}) \cdot \Delta(i/\sqrt{2}) \right| \right) \right)$$

and

$$B_{2,e_2} = - \left(24\zeta'(-1) + \log(8\pi^2) - \frac{11}{6}\log 2 + \frac{1}{12}\log \left(\left| \Delta(1/2 + i/2) \cdot \Delta(1 + i) \right| \right) \right).$$

Let us now consider the case when $k = 2$. From (43), we have that only e_1 and e_2 can be vanishing points of $E_4^{(2)}$. However, there are two possibilities: Either e_2 is an order two vanishing point, and $E_4^{(2)}(z) \neq 0$ for all $z \neq e_2$ in a fundamental domain \mathcal{F}_2 of X_2 , or e_1 is an order one vanishing point and $E_4^{(2)}(z) \neq 0$ for all points $z \neq e_1$ in \mathcal{F}_2 . If the latter possibility is true, then $E_6^{(2)}(z)/E_4^{(2)}(z)$ would be a weight 2 holomorphic modular form which vanishes only at e_2 , which is not possible since there is no weight two modular form on X_N for any squarefree N such that the surface X_N has genus zero; see [JST15]. Therefore, $E_4^{(2)}$ vanishes at e_2 of order two, and there are no other vanishing points of $E_4^{(2)}$ on X_2 .

Hence, in the notation of Theorem 9, we have $H_4^{(2)}(z) := H_{E_4^{(2)}}(z) = H_2(z, e_2)^2$, implying that

$$E_4^{(2)}(z) = C_{2,4}H_2(z, e_2)^2, \quad (45)$$

where $|C_{2,4}| = e^{2B_{2,e_2}}$. This proves that $H_2(z, e_2)^2$ is a weight four holomorphic modular function on $\overline{\Gamma_0(2)^+}$. If we combine (44) with (45) we get

$$H_2(z, e_1)^2 = \frac{C_{2,4}}{C_{2,6}^2} \cdot \frac{(E_6^{(2)}(z))^2}{E_4^{(2)}(z)};$$

in other words, $H_2(z, e_1)^2$ is a weight eight holomorphic modular function on $\overline{\Gamma_0(2)^+}$.

Furthermore, application of Proposition 11 with $f_{2k} = E_4^{(2)}$ and $Z_{f_{2k}} = \{e_2\}$ (with multiplicity two) together with Example 4 and the representation formula (12) yield (8).

By applying Proposition 11 with $f_{2k} = E_6^{(2)}$ and $Z_{f_{2k}} = \{e_1, e_2\}$ together with formula (8) we get the elliptic Kronecker limit formula for $\mathcal{E}_{e_1}^{\text{ell}}(z, s)$

$$\mathcal{E}_{e_1}^{\text{ell}}(z, s) = -s \log \left(|E_6^{(2)}(z)| |E_4^{(2)}(z)|^{-1/2} |\Delta(z)\Delta(2z)|^{-1/6} \right) + O(s^2) \text{ as } s \rightarrow 0.$$

Example 14. Consider the surface X_5 . There exist three order two elliptic elements, namely $e_1 = i/\sqrt{5}$, $e_2 = 2/5 + i/5$, and $e_3 = 1/2 + i/(2\sqrt{5})$. The surface X_5 has genus zero and one cusp, hence $\text{Vol}_{\text{hyp}}(X_5) = \pi$. Using the transformation rule for $E_6^{(5)}$, one concludes that the holomorphic form $E_6^{(5)}$ must vanish at e_1 , e_2 and e_3 . By the dimension formula (38), one sees that e_1 , e_2 and e_3 are the only zeros of $E_6^{(5)}$. Theorem 9 then implies that

$$E_6^{(5)}(z) = C_{5,6}H_6^{(5)}(z), \quad (46)$$

where the absolute value of the constant $C_{5,6}$ is given by $|C_5| = e^{B_{5,e_1} + B_{5,e_2} + B_{5,e_3}}$ and

$$B_{5,e_1} + B_{5,e_2} + B_{5,e_3} = -3(24\zeta'(-1) + \log(8\pi^2)) - \log 50 \\ + \frac{1}{12} \log \left(\left| \Delta(i/\sqrt{5})\Delta(i\sqrt{5})\Delta(2/5 + i/5)\Delta(2+i)\Delta(1/2 + i/(2\sqrt{5}))\Delta(5/2 + i\sqrt{5}/2) \right| \right).$$

One can view (46) as analogue of the Jacobi triple product formula.

Remark 15. Let $N = p_1 \cdot \dots \cdot p_r$ be a squarefree number. Then the surface X_N has one cusp. Numerous results are known concerning the topological structure of X_N ; see, for example, [Cum04] and references therein. As a consequence, one can develop a number of results similar to the above examples when $N = 2$ or $N = 5$. In particular, Theorem 9 holds, so one can factor any holomorphic Eisenstein series $E_{2k}^{(N)}$ of weight $2k$ into a product of elliptic Kronecker limit functions, up to a factor of modulus one.

6.3 Congruence subgroups of prime level

Consider the surface M_p for a prime p . The smallest positive integer k such that there exists a weight $2k$ holomorphic form is $k = 1$. As a result, we have the following corollary of Theorem 9.

Corollary 16. Let $f_{2k,p}$ denote weight $2k \geq 2$ holomorphic form on the surface $M_p = \overline{\Gamma_0(p)} \setminus \mathbb{H}$ bounded at cusps and such that the constant term in its q -expansion is equal to $b_{f_{2k,p}}$. Then,

$$f_{2k,p}(z) = a_{f_{2k,p}} \tilde{B}_{f_{2k,p}} \prod_{w \in Z(f_{2k,p})} \tilde{H}_p(z, w),$$

where $a_{f_{2k,p}}$ is a complex constant of modulus one and

$$\tilde{B}_{f_{2k,p}} = |b_{f_{2k,p}}| \prod_{w \in Z(f_{2k,p})} \left(\exp \left[-C_w \left(\frac{2p^2 \log p}{p^2 - 1} + C \right) \right] \right)^{p-1} \left| \frac{\eta(pw)^p}{\eta(w)} \operatorname{Im}(w) \right|^{-C_w}$$

with $C := \log(8\pi^2) + 24\zeta'(-1)$.

Let us now compute the constants $\tilde{B}_{f_{2k,p}}$ for two cases.

Example 17. If $p = 2$, then the surface M_2 has only one elliptic point, $e = 1/2 + i/2$, which has order two. Furthermore, $\operatorname{Vol}_{\text{hyp}}(M_p) = \pi$, hence formula (38) with $k = 1$ implies that the holomorphic form $E_{2,2}$ defined by (10) with $p = 2$ vanishes only at e , and the vanishing is to order one. From the q -expansion (11) we have that $|b_{E_{2,2,2}}| = 2 - 1 = 1$. Since $C_e = 1$, we get

$$E_{2,2}(z) = a_2 \cdot \frac{1}{16\sqrt[3]{4}\pi^2} \exp(-24\zeta'(-1)) \left| \frac{\eta(1/2 + i/2)}{\eta(1+i)^2} \right| \tilde{H}_2(z, e),$$

for some complex constant a_2 of modulus one. In other words, the elliptic Kronecker limit function $\tilde{H}_2(z, e)$ is a weight two modular form on $\overline{\Gamma_0(2)}$.

Example 18. If $p = 3$, then the surface M_3 has only one elliptic point $e = 1/2 + \sqrt{3}i/6$, which has order three. The volume of the surface M_3 is $4\pi/3$, hence formula (38) with $k = 1$ implies that the holomorphic form $E_{2,3}$ vanishes only at e , of order two. Furthermore, $|b_{E_{2,2,2}}| = 2$ and $C_e = 1/2$, so then

$$E_{2,3}(z) = a_3 \cdot \frac{1}{12\sqrt[4]{27}\pi^2} \exp(-24\zeta'(-1)) \left| \frac{\eta(1/2 + i\sqrt{3}/6)}{\eta(3/2 + i\sqrt{3}/2)^3} \right| \tilde{H}_3(z, e)^2,$$

for some complex constant a_3 of modulus one.

7 Additional considerations

In this section, we use the elliptic Kronecker's limit function to prove Weil's reciprocity law. In addition, we state various concluding remarks.

7.1 Weil reciprocity

To conclude this article, we will use equation (3) to prove Weil's reciprocity law which, for the convenience of the reader, we now state.

Theorem 19. [Weil Reciprocity] *Let f and g be meromorphic functions on the smooth, compact Riemann surface M . Let D_f and D_g denote the divisors of f and g , respectively, which we write as*

$$D_f = \sum m_f(P)P \quad \text{and} \quad D_g = \sum m_g(P)P.$$

Then

$$\prod_{w_j \in D_g} f(w_j)^{m_g(w_j)} = \prod_{z_i \in D_f} g(z_i)^{m_f(z_i)}.$$

Proof. Consider the function

$$I(s; f, g) = \sum_{z_i \in D_f} \sum_{w_j \in D_g} m_f(z_i) m_g(w_j) \mathcal{E}_{w_j}^{\text{ell}}(z_i, s).$$

We shall compute the asymptotic expansion of $I(s; f, g)$ near $s = 0$. Since both D_f and D_g have degree zero, we immediately have the equations

$$\sum_{z_i \in D_f} \sum_{w_j \in D_g} m_f(z_i) m_g(w_j) c = 0$$

and

$$\sum_{z_i \in D_f} \sum_{w_j \in D_g} m_f(z_i) m_g(w_j) \log((\text{Im}(z_i))^c) = 0.$$

Since M is assumed to be smooth and compact, the terms in (3) involving the parabolic Eisenstein series do not appear. Hence, we have the asymptotic expansion

$$I(s; f, g) = - \sum_{z_i \in D_f} \sum_{w_j \in D_g} m_f(z_i) m_g(w_j) \log(|H(z_i, w_j)|) \cdot s + O(s^2) \quad \text{as } s \rightarrow 0. \quad (47)$$

Weil's reciprocity formula will be proved by evaluating

$$\lim_{s \rightarrow 0} s^{-1} I(s; f, g)$$

in two different ways, one by first summing over the points in D_f the sum over the points in D_g , and the second way obtained by interchanging the order of summation.

To begin, we claim there exist constants a_f and a_g such that

$$f(w) = a_f \prod_{z_i \in D_f} H(z_i, w)^{m_f(z_i)} \quad \text{and} \quad g(z) = a_g \prod_{w_j \in D_g} H(z, w_j)^{m_g(w_j)}.$$

Indeed, both sides of each proposed equality are meromorphic functions with the same divisors, hence, differ by a multiplicative constant. Since both D_f and D_g have degree zero, one has that

$$\prod_{z_i \in D_f} |a_g|^{m_f(z_i)} = \prod_{w_j \in D_g} |a_f|^{m_g(w_j)} = 1.$$

Therefore, we can write the lead term in (47) in two ways, yielding the identity

$$\prod_{w_j \in D_g} |f(w_j)|^{m_g(w_j)} = \prod_{z_i \in D_f} |g(z_i)|^{m_f(z_i)}. \quad (48)$$

It remains to argue that (48) holds without the absolute value signs, which can be completed as follows. First, apply the above arguments in a fundamental domain \mathcal{F} of M whose interior contains the support of D_f and D_g . On such a domain, one can choose a well-defined branch of $H(z, w)$, hence we arrive at the equality

$$\prod_{w_j \in D_g} f(w_j)^{m_g(w_j)} = \prod_{z_i \in D_f} g(z_i)^{m_f(z_i)} \quad (49)$$

viewing all points z_i and w_j as lying in \mathcal{F} . Now, when tessellating by $\eta \in \Gamma$, one introduces multiplicative factors of the form

$$\prod_{w_j \in D_g} \epsilon_\Gamma(\eta)^{m_g(w_j)} \quad \text{and} \quad \prod_{z_i \in D_f} \epsilon_\Gamma(\eta)^{m_f(z_i)}. \quad (50)$$

Since D_f and D_g are degree zero, each term in (50) is equal to one. Therefore, one gets a well-defined extension of (49) to all $z, w \in \mathbb{H}$, which completes the proof of Theorem 19. \square

7.2 Unitary characters and Artin formalism

As with parabolic Eisenstein series, one can extend the study of elliptic Eisenstein series to include the presence of a unitary character. More precisely, let $\pi : \Gamma \rightarrow U(n)$ denote an n -dimensional unitary representation of the group Γ with associated character χ_π . Let us define

$$\mathcal{E}_w^{\text{ell}}(z, s; \pi) = \sum_{\eta \in \Gamma} \chi_\pi(\eta) \sinh(d_{\text{hyp}}(\eta z, w))^{-s} \quad (51)$$

to be the elliptic Eisenstein series twisted by χ_π . Note that if $n = 1$ and π is trivial, then the above definition is equal to $\text{ord}(w)$ times the series in (2). (Again, we kept the definition (2) in order to be consistent with the notation in [JvPS14]). In general terms, the meromorphic continuation of (51) can be studied using the methodology of [JvPS14], which depended on the spectral expansion and small time asymptotics of the associated heat kernel. As a result, we feel it is safe to say that one subsequently can prove the continuation of (51).

Having established the meromorphic continuation of (51), one then can study the elliptic Kronecker limit functions. It would be interesting to place the study in the context of the Artin formalism relations (see [JLa94] and references therein). The system of elliptic Eisenstein series associated to the representations π will satisfy additive Artin formalism relations, and, through exponentiation, the corresponding elliptic Kronecker limit functions will satisfy multiplicative Artin formalism relations. It would be interesting to carry out these computations in the setting of the congruence groups $\Gamma_0(N)$ as subgroups of the moonshine groups $\Gamma_0(N)^+$, for instance, in order to relate the above-mentioned computations for parabolic Kronecker limit functions. It is possible that a similar approach could yield further relations amongst the elliptic Kronecker limit functions.

7.3 The factorization theorem in other cases

7.3.1 Factorization for compact surfaces

If M is compact then, in a sense, Theorem 9 becomes the following. In the notation of the proof Theorem 9, the quotient

$$F_{f_{2k}}(z) := \frac{H_{f_{2k}}(z)}{f_{2k}(z)}$$

is a non-vanishing, bounded, holomorphic function on M , hence is constant, thus

$$f_{2k}(z) = c_{f_{2k}} H_{f_{2k}}(z) := c_{f_{2k}} \prod_{w \in Z(f_{2k})} H_\Gamma(z, w)$$

for some constant c_f . The point now is to develop a strategy by which one can evaluate $c_{f_{2k}}$. Perhaps the most natural approach would be to study the limiting value of

$$\tilde{H}_\Gamma(z) := \lim_{w \rightarrow z} \frac{H_\Gamma(z, w)}{z - w},$$

which needs to be considered in the correct sense as a holomorphic form on M . One can then express $c_{f_{2k}}$ in terms of the first non-zero coefficient of f_{2k} about a point $z \in Z(f_{2k})$, a product of the forms $H_{f_{2k}}(z, w)$ for two different points in $Z(f_{2k})$ and $\tilde{H}_\Gamma(z)$. Such formulae could be quite interesting in various cases of arithmetic interest. We will leave the development of such identities for future investigation.

7.3.2 Factorization for surfaces with more than one cusp

It is evident that one can generalize Theorem 9 to the case when the holomorphic form f_{2k} vanishes in a cusp, or several cusps. In such an instance, one includes factors of the parabolic Kronecker limit function in the construction of $H_{f_{2k}}$. The parabolic Kronecker limit function is bounded and non-vanishing in any cusp other than the one to which it is associated, and the (fractional) order to which it vanishes follows from Theorem 1 of [Ta86]. As with Theorem 9, one can express any holomorphic modular form as a product of parabolic and elliptic Kronecker limit functions, up to a multiplicative constant. Furthermore, the multiplicative constant can be computed, up to a factor of modulus one, from the value of the various functions at a cusp.

References

- [SCM66] *Seminar on complex multiplication*, eds. A. Borel, S. Chowla, C. S. Herz, K. Iwasawa, J. P. Serre, Lecture Notes in Mathematics **21** Springer-Verlag, Berlin-New York, 1966.
- [Cum04] Cummins, C.: *Congruence subgroups of groups commensurable with $\mathrm{PSL}(2, \mathbb{Z})$ of genus 0 and 1*. Experiment. Math. **13** (2004), 361–382.
- [De91] Deligne, P.: Le symbole modéré. Inst. Hautes Études Sci. Publ. Math. No. **73** (1991), 147–181.
- [Fa07] Falliero, T.: *Dégénérescence de séries d'Eisenstein hyperboliques*. Math. Ann. **339** (2007), 341–375.
- [GJM08] Garbin, D., Jorgenson, J. and Munn, M.: *On the appearance of Eisenstein series through degeneration*. Comment. Math. Helv. **83** (2008), 701–721.
- [GvP09] Garbin, D. and von Pippich, A.-M.: *On the behavior of Eisenstein series through elliptic degeneration*. Comm. Math. Phys. **292** (2009), 511–528.
- [Go73] Goldstein, L. J.: *Dedekind sums for a Fuchsian group I*, Nagoya Math. J. **80** (1973), 21–47.
- [GH78] Griffiths, P. and Harris, J.: *Principles of algebraic geometry*. John Wiley & Sons, New York, 1978.
- [He83] Hejhal, D.: *The Selberg trace formula for $\mathrm{PSL}(2, \mathbb{R})$. II*, Lecture Notes in Math. **1001**, Springer-Verlag, Berlin, 1983.
- [Hel66] Helling, H.: Bestimmung der Kommensurabilitätsklasse der Hilbertschen Modulgruppe. Math. Z. **92** (1966), 269–280.
- [Iwa02] Iwaniec, H.: *Spectral methods of automorphic forms*. Graduate Studies in Mathematics **53**, American Mathematical Society, Providence, RI, 2002.
- [JKvP10] Jorgenson, J., Kramer, J. and von Pippich, A.-M.: *On the spectral expansion of hyperbolic Eisenstein series*. Math. Ann. **346** (2010), 931–947.
- [JLa94] Jorgenson, J. and Lang, S.: *Artin formalism and heat kernels*. J. Reine Angew. Math. **447** (1994) 165–200.
- [JvPS14] Jorgenson, J., von Pippich, A.-M. and Smajlović, L.: *On the wave representation of hyperbolic, elliptic, and parabolic Eisenstein series*, submitted for publication.

- [JST14] Jorgenson, J, Smajlović, L. and Then, H.: *On the distribution of eigenvalues of Maass forms on certain moonshine groups*. Math. Comp. **83** (2014), 3039–3070.
- [JST14a] Jorgenson, J, Smajlović, L. and Then, H.: *Kronecker’s limit formula, holomorphic modular functions and q -expansions on certain moonshine groups*, submitted for publication. preprint.
- [JST15] Jorgenson, J, Smajlović, L. and Then, H.: *Aspects of holomorphic function theory on genus zero moonshine groups*, in preparation.
- [Kh08] Khovanskii, A.: Logarithmic functional and reciprocity laws. In: *Toric topology*, M. Harada, Y. Karshon, M. Masuda, and T. Panov, eds. Contemp. Math., **460**, Amer. Math. Soc., Providence, RI, (2008), 221–229.
- [Ku73] Kubota, T.: *Elementary theory of Eisenstein series*. Kodansha Ltd., Tokyo, 1973.
- [KM79] Kudla, S. S. and Millson, J. J.: *Harmonic differentials and closed geodesics on a Riemann surface*. Invent. Math. **54** (1979), 193–211.
- [La76] Lang, S. *Introduction to modular forms*. Grundlehren der mathematischen Wissenschaften **222**, Springer-Verlag, Berlin, 1976.
- [La88] Lang, S.: *Introduction to Arakelov theory*. Springer-Verlag, Berlin, 1988.
- [vP10] von Pippich, A.-M.: *The arithmetic of elliptic Eisenstein series*. PhD thesis, Humboldt-Universität zu Berlin, 2010.
- [vP15] von Pippich, A.-M.: *A Kronecker limit type formula for elliptic Eisenstein series*, in preparation.
- [Se73] Serre, J.-P.: *A Course in Arithmetic*, Graduate Texts in Mathematics, **7**, Springer-Verlag, New York, 1973.
- [Sh71] Shimura, G.: *Introduction to the Arithmetic Theory of Automorphic Forms*, Publications of the Mathematical Society of Japan, Princeton University Press, Princeton, 1971.
- [Si80] Siegel, C. L.: *Advanced analytic number theory*. Tata Institute of Fundamental Research Studies in Mathematics, **9**, Tata Institute of Fundamental Research, Bombay, 1980.
- [Ta86] Takada, I.: *Dedekind sums of $\Gamma(N)$* . Japan J. Math. **12** (1986), 401–411.
- [Va96] Vassileva, I. N.: *Dedekind eta function, Kronecker limit formula and Dedekind sum for the Hecke group*, PhD thesis, University of Massachusetts Amherst, 1996.

Jay Jorgenson
 Department of Mathematics
 The City College of New York
 Convent Avenue at 138th Street
 New York, NY 10031 U.S.A.
 e-mail: jjorgenson@mindspring.com

Anna-Maria von Pippich
 Fachbereich Mathematik
 Technische Universität Darmstadt
 Schloßgartenstr. 7
 D-64289 Darmstadt
 Germany
 e-mail: pippich@mathematik.tu-darmstadt.de

Lejla Smajlović
Department of Mathematics
University of Sarajevo
Zmaja od Bosne 35, 71 000 Sarajevo
Bosnia and Herzegovina
e-mail: lejlas@pmf.unsa.ba